Eidgenössische
Technische Hochschule
Zürich

Ecole polytechnique fédérale de Zurich
Politecnico federale di Zurigo
Federal Institute of Technology at Zurich

Departement of Computer Science
Markus Püschel, David Steurer

3. Dezember 2018

Algorithms & Data Structures — Homework 11 — HS 18

Exercise Class (Room & TA): _____

Submitted by: _____

Peer Feedback by: _____

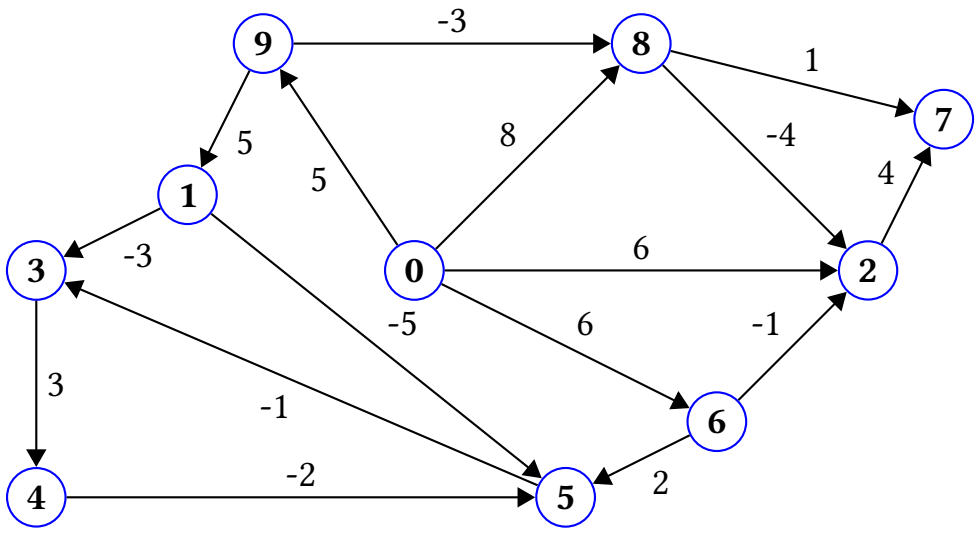
Points: _____

Hint: This exercise sheet is concerned with *dynamic programming*. A complete description of a dynamic program always includes the following aspects (important also for the exam!):

1. **Definition of the DP table:** What are the dimensions of the dynamic programming table DP[., .]? What is the meaning of each entry (in clearly worded words)?
2. **Calculation of an entry:** Which values of the table are initialized, and how are they initialized? How are entries calculated from other entries? What are the dependencies between entries?
3. **Calculation order:** In what order can you calculate the entries so that these dependencies are fulfilled?
4. **Reading the solution:** How can the solution be read out from the table at the end?

Exercise 11.1 *Bellman-Ford.*

Consider the following graph $G = (V, E)$ with edge weights $w\colon E \to \mathbb{R}$. Execute the Bellman–Ford algorithm on this graph with source node $s = 0$. Compute the values $T(j, \ell)$ for all $j \in V$ and all $\ell \in \{0, 1, \ldots, n-1\}$ as described in class. Use arrows to indicate for every entry $T(j, \ell)$ computed via the recurrence equation which entry in the previous row determined the value of $T(j, \ell)$.



Exercise 11.2 *Arbitrage* **(1 Point for 1).**

When trading currencies, *arbitrage* means to exploit price differences in order to profit by exchanging currencies multiple times. For example, on June 2nd, 2009, 1 US Dollar could be exchanged for 95.729 Yen, 1 Yen for 0.00638 Pound sterling, and 1 Pound sterling for 1.65133 US Dollars. If a trader exchanged 1 US Dollar for Yen, exchanged the obtained amount for Pound sterling and finally exchanged this amount back to US Dollars, he would have obtained $95.729 \cdot 0.00638 \cdot 1.65133 \approx 1.0086$ US Dollars, corresponding to a gain of 0.86%.

1. You are given n currencies $\{1, \ldots, n\}$ and an $(n \times n)$ exchange rate matrix $R \in (\mathbb{Q}^+)^2$. For two currencies $i, j \in \{1, \ldots, n\}$ one unit of currency i can be exchanged for $R(i, j) > 0$ units of currency j. The goal is to decide whether an arbitrage opportunity exists, i.e., if there exists a sequence of k different currencies $W_1, \ldots, W_k \in \{1, \ldots, n\}$ such that $R(W_1, W_2) \cdot R(W_2, W_3) \cdots R(W_{k-1}, W_k) \cdot R(W_k, W_1) > 1$ holds.

   Model the above problem as a graph problem. Show how the input can be transformed into a directed, weighted graph $G = (V, E, w)$ that contains a cycle with negative weight *if and only if* an arbitrage activity is possible. Justify why G contains a negative cycle if and only if an arbitrage opportunity exists.

   Notice: Using logarithms might be beneficial because of the property $\ln(a \cdot b) = \ln(a) + \ln(b)$.

2. Show that the Bellman-Ford algorithm can be used to detect negative cycles. Recall that this algorithm computes values $T(j, \ell)$, the minimum weight of an s-j walk with at most ℓ edges. Show that a graph contains a negative cycle reachable from s if and only if there exists a node j such that $T(j, n-1) \neq T(j, n)$, where n is the number of vertices. (In class, we only argued about the "if" direction but not the "only if" direction.)

3. Use the previous two parts of this exercise to design an algorithm that decides if an arbitrage opportunity exists. What is the best running time you can get (in terms of n)?

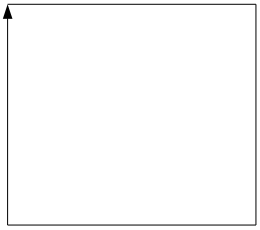
Exercise 11.3 *Side Gig* **(2 Points).**

To earn cash, you decide to take a second job as driver for a ride hailing service. As a driver, your job is to drive from location to location, picking up passengers and dropping them off.

You decide to use the following rules to model the problem:

- In order to pick up or drop off a passenger at a location, you must be on the same block as the location. (Being on the same block as the location means that you and the location are both located on the same street, between two adjacent intersections)
- You drive from intersection to intersection. At an intersection you can:
  - Go straight. This takes 2 minutes.
  - Turn right. This takes 1 minute.
  - Turn left. This takes 3 minutes.
  - Take a U-turn. This reverses your direction, takes you to the opposite side of the street, and takes 4 minutes.
- You drive on the right side of the road.
- There are one-way streets, and you cannot turn or go straight onto a one way street going the wrong way.

1. Model the problem using a graph and design a solution that allows you to find the shortest path from location to location. Draw the resulting graph for the following street map.



2. When turning left or taking a U-turn, you have to wait at the intersection for a long time, wasting gasoline and emitting greenhouse gases. Therefore, you have decided that you will take at most L such turns during each trip, where $L \geq 1$.

   Model the problem using a graph and design a solution that allows you to find the shortest path from location to location.

Submission: On Monday, 10.12.2018, hand in your solution to your TA *before* the exercise class starts.